UNIT - 3

Stationary process:

A stochastic process $\{x(t), t \in T\}$ is a stationary process if finite dimensional dist. are invarient under arbitrary translation of the time parameter (ie.,) if $\forall$ 'n',

$$P\{x(t_1) \le x_1 ; x(t_2) \le x_2 \dots x(t_n) \le x_n\} = P\{x(t_1 + h) \le x_1 ; x(t_2 + h) \le x_2 \dots x(t_n + h) \le x_n\} \rightarrow (1)$$

for $t_i \in T$, $t_i + h \in T$, $h > 0$, then $\{x(t), t \ge 0\}$ is a stationary process (or) complete stationary (or) strict stationary; where $x(t)$ represents the random component of time series.

$$y(t) = f(t) + x(t)$$

where $f(t)$ is the systematic part & it is represented by a deterministic fn/- of time. The components of systematic part are trend, cyclical & seasonal component. Model representing random part are discussed for stationary stochastic process.

The process is a stationary of order 'n' equ (1) holds for some particular 'n'. The stochastic process is a covariance stationary or ~~covariance~~ wide-sense stationary or weak-sense stationary if the covariance function

$$\text{cov}[X(t), X(t+h)] = E[X(t).X(t+h)] - E[X(t)].E[X(t+h)]$$

depends only on 'h' and $\forall\, t \in T$.

Ex : 1

Let $\{X(t), t \geq 0\}$ be the stochastic process with p.m.f.

$$P\{X(t) = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \quad ; \lambda > 0, n = 0, 1, 2, \ldots$$

Soln:

$$E[X(t)] = \lambda t$$

$$V[X(t)] = \lambda t$$

$\Rightarrow$ Mean and variance depends on 't'.

Hence the process is not a stationary but it is evolutionary process.

Ex: 2

Let $\{x(t), t \geq 1\}$ be the stochastic process having mean 0 and var 1. Further $\{x_n\}$ be the seq of independent r.v's.

<u>Soln:</u>

$$E[x(t)] = 0$$

$$V[x(t)] = 1 \text{ is indept. of } 't' ..$$

$\Rightarrow x(t)$ is a stationary process.

$$\text{cov}[x(t), x(t+h)] = E[x(t). x(t+h)] - E[x(t)]. E[x(t+h)].$$

$$= E[x(t). x(t+h)] - 0$$

$$= \begin{cases} V[x(t)] & \text{if } h = 0. \\ 0 & \text{if } h \neq 0 \end{cases}$$

$$\text{cov}[x(t), x(t+h)] = \begin{cases} 1 & \text{if } h = 0 \\ 0 & \text{if } h \neq 0. \end{cases}$$

$\Rightarrow \text{cov}[x(t), x(t+h)]$ is an indept. of 't'.

Hence $\{x(t), t \geq 1\}$ is a covariance stationary (or) weak sense stationary.

Ex:3

Let us consider the stochastic process $\{x(t), t \geq 0\}$ where $X(t) = A_1 + A_2 t$; where A_1, A_2 are indept., r.v's with

$E(A_i) = \alpha_i$, $V(A_i) = \sigma_i^2$, for $i = 1, 2$.

Soln:

Given that,

$$x(t) = A_1 + A_2 t,$$

where A_1, A_2 are indept. r.v's with

$$E(A_1) = \alpha_1$$
$$E(A_2) = \alpha_2$$
$$V(A_1) = \sigma_1^2$$
$$V(A_2) = \sigma_2^2$$
$$x(t) = A_1 + A_2 t$$

Taking expectation on both sides,

$$E[x(t)] = E[A_1 + A_2 t]$$
$$= E(A_1) + t\,E(A_2)$$
$$E[x(t)] = \alpha_1 + t\alpha_2 \rightarrow (1)$$
$$V[x(t)] = E[x(t)]^2 - \{E[x(t)]\}^2$$

Consider $E[x(t)]^2 = E[A_1 + A_2]^2$

$$(V(x) = E(x^2) - [E(x)]^2$$
$$V(x) + [E(x)]^2 = E(x^2)$$
$$\sigma_1^2 + \alpha_1^2 = E(x^2))$$

$E[x(t)]^2 = E[A_1 + A_2 t]^2$

$= E[A_1^2 + A_2^2 t^2 + 2tA_1A_2]$

$= E(A_1^2) + t^2(A_2^2) + 2t\, E(A_1A_2)$

$= [\sigma_1^2 + \alpha_1^2] + t^2[\sigma_2^2 + \alpha_2^2] + 2t\alpha_1\alpha_2.$

$= [\sigma_1^2 + t^2\sigma_2^2] + [\alpha_1^2 + t^2\alpha_2^2] + 2t\alpha_1\alpha_2$

$V[x(t)] = E[x(t)]^2 - \{E[x(t)]\}^2$

$= [\sigma_1^2 + t^2\sigma_2^2 + (\alpha_1^2 + t^2\alpha_2^2) + 2t\alpha_1\alpha_2] - [\alpha_1 + \alpha_2 t]^2$

$= [\sigma_1^2 + t^2\sigma_2^2 + \cancel{\alpha_1^2} + \cancel{t^2\alpha_2^2} - \cancel{\alpha_1^2} - \cancel{t^2\alpha_2^2} + 2\alpha_1\alpha_2 t - 2t\alpha_1\alpha_2$

$= \sigma_1^2 + t^2\sigma_2^2.$

$\therefore V[x(t)] = \sigma_1^2 + t^2\sigma_2^2.$

Since mean and variance are the func. of 't',

the process is not stationary. Then the

covariance fn /-

$Cov[x(t).x(s)] = E[x(t).x(s)] - E[x(t)].E(x(s))]$

Then,

$E[x(t).x(s)] = E[(A_1 + A_2 t)(A_1 + A_2 s)].$

$= E[A_1^2 + A_1A_2 s + A_1A_2 t + A_2^2 ts]$

$= E[A_1^2] + sE[A_1A_2] + tE[A_1A_2] + ts\,E(A_2^2)$

$= (\sigma_1^2 + \alpha_1^2) + s(\alpha_1\alpha_2) + t(\alpha_1\alpha_2) + ts(\sigma_2^2 + \alpha_2^2)$

$\therefore \text{cov}\,[X(t)\cdot X(s)] = \sigma_1^2 + \alpha_1^2 + s\alpha_1\alpha_2 + t\alpha_1\alpha_2$

$+ ts(\sigma_2^2 + \alpha_2^2) - \{(\alpha_1 + t\alpha_2)(\alpha_1 + s\alpha_2)\}$

$= \sigma_1^2 + \alpha_1^2 + s\alpha_1\alpha_2 + t\alpha_1\alpha_2 + ts(\sigma_2^2 + \alpha_2^2) -$

$[\alpha_1^2 + \alpha_1\alpha_2 s + \alpha_1\alpha_2 t + ts\alpha_2^2]$

$= \sigma_1^2 + \cancel{\alpha_1^2} + s\cancel{\alpha_1\alpha_2} + t\cancel{\alpha_1\alpha_2} + ts\sigma_2^2 + ts\cancel{\alpha_2^2} -$

$\cancel{\alpha_1^2} - \cancel{\alpha_1\alpha_2}s - \cancel{\alpha_1\alpha_2}t - ts\cancel{\alpha_2^2}$

$= \sigma_1^2 + ts\sigma_2^2.$

It depends on the time parameter t, s. Hence, the covariance depends on t and s. The process is not covariance stationary but is an evolutionary process.

Ex: 4

Examine whether $\{X(t), t \geq 0\}$ is a stationary process or evolutionary process, where $X(t) = A\cos\omega t + B\sin\omega t$, where A and B are uncorrelated r.v's with mean 0 and var 1. ω is a +ve constant.

Soln:

Given that,

$X(t) = A\cos\omega t + B\sin\omega t,$

Further given that $\operatorname{cov}(AB) = 0$

$E(A) = 0 = E(B)$, $V(A) = V(B) = 1$

$\therefore E[x(t)] = E[A \cos wt + B \sin wt]$

$= E[A \cos wt] + E[B \sin wt]$

$= \cos wt \cdot E(A) + \sin wt\, E(B)$.

$E[x(t)] = 0 + 0 = 0$.

$V[x(t)] = E[A \cos wt + B \sin wt]^2$

$= E[A^2 \cos^2 wt + B^2 \sin^2 wt + 2AB \cos wt \sin wt]$

$= \cos^2 wt\, E(A^2) + \sin^2 wt\, E(B^2) + 2 \cos wt \sin wt\, E(AB)]$

$= \cos^2 wt\,(1) + \sin^2 wt\,(1) + 2 \cos wt \sin(0)$

$= \cos^2 wt + \sin^2 wt = 1$.

Stationary process, It is indepl, of t.

Here, $E[x(t)] = 0$, $V[x(t)] = 1$

$\operatorname{cov}[x(t), x(s)] = E[(A \cos wt + B \sin wt)$

$(A \cos ws + B \sin ws)]$

$= E[A^2 \cos wt \cos ws + AB \cos wt \sin ws$

$+ B \sin wt\, A \cos ws + B^2 \sin wt \sin ws]$

$= E(\overset{1+0}{A^2}) \cos wt \cos ws + E(\overset{0}{AB}) \cos wt \cdot \sin ws$

$+ E(\overset{0}{AB}) \cos ws \sin wt + E(\overset{1+0}{B^2}) \sin wt \sin ws$.

$= \cos wt \cos ws + \sin wt \sin ws.$

$= \cos(s-t)w.$ -(formula).

$\Rightarrow cov[x(t), x(s)] = \cos(s-t)w.,$

depends on $(s-t)$. Hence $x(t)$ is not a cov. stationary.

Definition:

A stationary process $\{x(t), t \in T\}$ is said to be weakly stationary or stationary in wide-sense if the following are satisfied,

i) $E[x_t] = \mu \leq \infty$

ii) $E[x_t^2] \leq \alpha$ exists.

iii) $cov(s,t) = cov(t-s).$

Ex:

Let us consider the process $x(t)$ such that,

$$P\{x(t) = n\} = \frac{(at)^{n-1}}{(1+at)^{n+1}}, \forall n = 1, 2, 3, \ldots \infty.$$

Soln:

$E(x) = \Sigma x\, p(x)$

$$E[x(t)] = \sum_{n=1}^{\infty} [x=n] \cdot p[x(t)=n].$$

$$= \sum_{n=1}^{\infty} n \cdot \frac{(at)^{n-1}}{(1+at)^{n+1}}$$

$$E[x(t)] = \sum_{n=1}^{3} n \frac{(at)^{n-1}}{(1+at)^{n-1+2}}$$

$$= \frac{1}{(1+at)^2} \sum_{1}^{3} n \frac{(at)^{n-1}}{(1+at)^{n-1}}$$

$$= \frac{1}{(1+at)^2} \left[1 + 2\left(\frac{at}{1+at}\right)^1 + 3\left(\frac{at}{1+at}\right)^2\right]$$

$$= \frac{1}{(1+\cancel{at})^2} (1+\cancel{at})^2 = 1$$

ie $E[x(t)] = 1$

$$V[x(t)] = E[x(t)]^2 - \{E[x(t)]\}^2$$

$$E[x(t)]^2 = \sum_{1}^{3} n^2 \frac{(at)^{n-1}}{(1+at)^{n+1}}$$

$$= \sum_{1}^{3} n(n-1) \frac{(at)^{n-1}}{(1+at)^{n+1}} + \sum_{1}^{3} n \frac{at}{(1+at)^{n+1}}$$

$$F \frac{at}{(1+at)^3} \sum_{n=2}^{3} \left[\frac{n(n-1)(at)^{n-2}}{(1+at)^{n-2}}\right] + \sum_{1}^{3} n \frac{(at)^{n-1}}{(1+at)^{n+1}}$$

$$\left\{ \because n^2 = n^2 - n + n = n(n-1) + n \right\}$$

$$= \frac{at}{(1+at)^3} \left[\sum_{n=2}^{3} \frac{n(n-1)(at)^{n-2}}{(1+at)^{n-2}}\right] + 1$$

$$= \frac{at}{(1+at)^3} \left[2 + (3 \times 2) \frac{at}{1+at}\right] + 1$$

$$E[x(t)]^2 = \frac{2at}{(1+at)^3}\left[1+3\left(\frac{at}{1+at}\right)\right]+1$$

$$= \frac{2at}{\cancel{(1+at)^3}} \cdot \cancel{(1+at)^3} + 1$$

$$= 2at + 1$$

$$V[x(t)] = E[x(t)]^2 - \{E[x(t)]\}^2$$

$$= 2at + \cancel{1} - \cancel{1}$$

$$= 2at \text{, which depends on 't'}$$

Hence $x(t)$ is not stationary.

Ex:

Let $x(t) = A_0 + A_1 t + A_2 t^2$

where A_0, A_1, A_2 are uncorrelated with mean zero and variance 1. Find the mean and cov. fn. and state whether the process is stationary or not.

Soln

Given that

$$x(t) = A_0 + A_1 t + A_2 t^2$$

$$E(A_0) = E(A_1) = E(A_2) = 0$$

$$V(A_0) = V(A_1) = V(A_2) = 1$$

$$\text{cov}(A_i, A_j) = 0 \; ; \; i = 0,1,2$$
$$j = 0,1,2 \; ; \; i \neq j$$

Then,

$$E[x(t)] = E[A_0 + A_1 t + A_2 t^2]$$

$$= E(A_0) + t\,E(A_1) + t^2 E(A_2)$$

$$= 0$$

$$V[x(t)] = E[x(t)]^2 - \{E[x(t)]\}^2$$

$$= E[x(t)]^2 - 0$$

$$E[x(t)]^2 = E[A_0 + A_1 t + A_2 t^2]^2$$

$$= E(A_0)^2 + t^2 E(A_1^2) + t^4 E(A_2^2)$$

$$E[x(t)]^2 = 1 + t^2 + t^4$$

$$\therefore V[x(t)] = 1 + t^2 + t^4$$

$$\boxed{\begin{aligned} V(x) &= E(x^2) - [E(x)]^2 \\ E(x^2) &= V(x) + [E(x)]^2 \\ &= 1 + 0 \\ &= 1 \end{aligned}}$$

$$\text{cov}[x(t), x(s)] = E[(A_0 + A_1 t + A_2 t^2)(A_0 + A_1 s + A_2 s^2)] - 0$$

$$\boxed{\because\ \text{cov}(x,y) = E(x,y) - E(x)E(y)}$$

$$= E[A_0^2 + A_0A_1 s + A_0A_2 s^2 + A_0A_1 t + A_1^2 ts + A_1A_2 ts^2 + A_0A_2 t^2 + A_1A_2 t^2 s + A_2^2 t^2 s^2]$$

$$= E(A_0^2) + s\,E(A_0A_1) + s^2 E(A_0A_2) + t\,E(A_0A_1) + ts\,E(A_1^2) + ts^2 E(A_1A_2) + t^2 E(A_0A_2) + t^2 s\,E(A_1A_2) + t^2 s^2 E(A_2^2)$$

$$= 1 + ts + t^2 s^2$$

Since $V[x(t)]$ and $\text{cov}[x(t), x(s)]$ are the fn of 't' the process $x(t)$ is not stationary.

It is also not cov. stationary.

Models for the Generation of Random part of Time series :

(i) purely Random process (White Noise process)

Let $Y(t) = f(t) + x(t)$ be the time series, where $f(t)$ represents systematic part which includes trend, seasonal and cyclical components.

$x(t)$ represents the random part of the time series. A completely random process $\{x(t), t \in T\}$ has mean

$$E[x(t)] = \mu$$

$$V[x(t)] = \sigma^2$$

& the cov. fn. $C_k = E[x(t) \cdot x(t+k)] = 0 \ \forall k \neq$

$$C_k = 0$$

$\Rightarrow$ The process $x(t)$ is a cov. stationary

μ & σ^2 are indep. of 't'. Then the process is said to be covariance stationary

(ii) Auto Regressive Process - Definition :

First order Markov process (or) Auto-Regressive process of order 1 [AR(1)] :

The process $\{x(t), t \in T\}$ has the structure $x(t) = a\, x(t-1) + \varepsilon_t, \ t = 1, 2, \ldots$ — ①

where ε_t is purely random process such that $E(\varepsilon_t) = 0$; $V(\varepsilon_t) = \sigma_e^2 = \sigma^2$

① becomes,

$$x(t) - a\, x(t-1) = \varepsilon_t \quad \text{——— ②}$$

Multiply ② by $x(t-h)$,

$$x(t)\,x(t-h) - a\,x(t-1)\,x(t-h) = \varepsilon_t\,x(t-h)$$

Taking expectations on both sides, we get

$$E[x(t).x(t-h)] - a\,E[x(t-1)\,x(t-h)] = E[\varepsilon_t\,x(t-h)]$$

which is called as Yule-walker equations.

for $h=0$,

$$E[x(t)\,x(t-0)] - a\,E[x(t-1).x(t-0)] = E[\varepsilon_t\,x(t-0)]$$

$$\sigma_x^2 - a\rho(1)\sigma_x^2 = \sigma^2$$

$$\sigma_x^2[1-a\rho(1)] = \sigma^2$$

$$\sigma_x^2 = \frac{\sigma^2}{1-a\rho(1)} \quad \text{—— ③}$$

for $h=1$,

$$E[x(t).x(t-1)] - a\,E[x(t-1)\,x(t-1)] = E[\varepsilon_t\,x(t-1)]$$

$$\rho(1)\sigma_x^2 - a\,\rho(0)\,\sigma_x^2 = 0$$

$$\sigma_x^2[\rho(1) - a\rho(0)] = 0$$

$$\rho(1) - a\rho(0) = 0$$

$$\rho(1) = a\rho(0)$$

$$\rho(1) = a \quad \{\because \rho(0)=1$$

Substitute $\rho(1)$ in ③,

$$\sigma_x^2 = \frac{\sigma^2}{1-a^2}$$

for $h = 2$,

$$E[x(t).x(t-2)] - aE[x(t-1)x(t-2)] = E[\varepsilon_t\, x(t-2)]$$

$$\rho(2)\sigma_x^2 - a\rho(1)\sigma_x^2 = 0$$

$$\sigma_x^2[\rho(2) - a\rho(1)] = 0$$

$$[\rho(2) - a\rho(1)] = 0$$

$$\rho(2) = a\rho(1)$$

$$\rho(2) = a.a = a^2$$

$$\vdots$$

for $h = n$, we get

$$\rho(n) = a^n$$

$\Rightarrow x(t)$ is covariance stationary.

Moving Average Process:

2m [The process $\{x(t), t \in T\}$ is represented as

$$x(t) = a_0\varepsilon_t + a_1\varepsilon_{t-1} + \dots + a_h\varepsilon_{t-h} \quad \text{——} (1)$$

where a_i's are real constants and $\{\varepsilon_t\}$ is a purely random process such that $E(\varepsilon_t) = 0$ & $V(\varepsilon_t) = \sigma^2$.

$x(t)$ can also be represented as

$$x(t) = \phi(B).\varepsilon_t$$

where $\phi(B) = \sum_{r=0}^{h} a_r B^r$

B = Backshift operator

ie $B[x(t)] = x(t-1)$

$B^2[x(t)] = x(t-2), \ldots$

when $a_h \neq 0$, $\{x(t), t \in T\}$ is called as moving average process of order 'h'.

$$E[x(t)] = E[\phi(B)\varepsilon_t]$$

$$= \phi(B)\, E(\varepsilon_t) = 0$$

The cov. fn. $c_k = E[x(t) \cdot x(t-k)]$

$$c_k = E\Big\{\big[a_0 \varepsilon_t + a_1 \varepsilon_{t-1} + \cdots + a_k \varepsilon_{t-k} + a_{k+1} \varepsilon_{t-k-1} + \cdots + a_h \varepsilon_{t-h}\big]$$

$$\cdot \big[a_0 \varepsilon_{t-k} + a_1 \varepsilon_{t-k-1} + \cdots + a_{h-k} \underset{t-k-(h-k)}{\varepsilon_{t-h}} + \cdots + a_h \varepsilon_{t-k-h}\big]\Big\}$$

$$C_k = E\Big[a_0 a_k \varepsilon_{t-k}^2 + a_1 a_{k+1} \varepsilon_{t-k-1}^2 + \cdots + a_h a_{h-k} \varepsilon_{t-h}^2 + \sum_{r \neq t} a_r a_t \varepsilon_r \varepsilon_t\Big]$$

$$= a_0 a_k E(\varepsilon_{t-k}^2) + a_1 a_{k+1} E(\varepsilon_{t-k-1}^2) + \cdots + a_h a_{h-k} E(\varepsilon_{t-h}^2)$$

$$\Rightarrow c_k = \begin{cases} (a_0 a_k + a_1 a_{k+1} + \cdots + a_h a_{h-k})\sigma^2 & ; k \leq h \\ 0 & ; k > h \end{cases}$$

The necessary condition for representing the stochastic process by moving average of order 'h' is that $c_k = 0$ for $k > h$

$$\therefore \rho_k = \frac{c_k}{c_0} = \begin{cases} \dfrac{a_0 a_k + a_1 a_{k-1} + \dots + a_{h-k} a_h}{a_0^2 + a_1^2 + \dots + a_h^2} & ; \ k \le h \\ 0 & ; \ k > h \end{cases}$$

$\Rightarrow x(t)$ is a cov. stationary.

Note :

If $a_k = \frac{1}{h+1}$, $\forall \ k = 0, 1, \dots h$

then $\rho = \frac{h-k+1}{h+1} = \frac{h+1-k}{h+1}$

$$\rho = \begin{cases} 1 - \dfrac{k}{h+1} & ; \ 0 \le k \le h \\ 0 & ; \ \text{o.w} \end{cases}$$

The graph of the corr. fn. is called as correlogram.

Auto-Regressive Process of order 2 [AR(2)]:

The process $x(t)$ is said to be second order Auto-Regressive if it satisfies the difference eqn,

$$x(t) + a_1 x(t-1) + a_2 x(t-2) = \varepsilon_t \quad \forall \ t \ge 0 \quad \text{——— (1)}$$

where a_1 & a_2 are constants, ε_t is a purely random process

By using backward shifting operator 'B', eqn (1) can be written as

$$B^0 x(t) + a_1 B^1 x(t) + a_2 B^2 x(t) = \varepsilon_t$$

$$\left[1 + a_1 B + a_2 B^2\right] x(t) = \varepsilon_t$$

Assume that μ_1 & μ_2 are the distinct roots of the characteristic eqn. of the form

$$\phi(z) = z^2 + a_1 z + a_2 = 0 \quad \text{——} \quad (2).$$

neglecting for large t, we get

$$(1-\mu_1 B)(1-\mu_2 B)\, x(t) = \varepsilon_t \quad \text{——} \quad (3).$$

$$x(t) = \frac{\varepsilon_t}{(1-\mu_1 B)(1-\mu_2 B)}$$

$$= \left[(1-\mu_1 B)(1-\mu_2 B)\right]^{-1} \varepsilon_t$$

$\times$ & $\div$ by $(\mu_1 - \mu_2)$ on RHS,

$$x(t) = \frac{1}{\mu_1 - \mu_2}\left[\frac{\mu_1 - \mu_2}{(1-\mu_1 B)(1-\mu_2 B)}\right] \varepsilon_t$$

$$= \frac{1}{\mu_1 - \mu_2}\left[\frac{\mu_1(1-\mu_2 B) - \mu_2(1-\mu_1 B)}{(1-\mu_1 B)(1-\mu_2 B)}\right] \varepsilon_t$$

$$\left\{\because \mu_1 - \mu_1 \cancel{\mu_2 B} - \mu_2 + \cancel{\mu_1 \mu_2 B} = \mu_1 - \mu_2\right.$$

$$= \frac{1}{\mu_1 - \mu_2}\left[\frac{\mu_1}{1-\mu_1 B} - \frac{\mu_2}{1-\mu_2 B}\right] \varepsilon_t$$

On expanding, we get

$$x(t) = \frac{1}{\mu_1 - \mu_2} \sum_{s=0}^{\infty} \left(\mu_1^{s+1} - \mu_2^{s+1}\right) \varepsilon_{t-s} \quad \text{——} \quad (4)$$

Let us consider the eqn (1)

$$x(t) + a_1 x(t-1) + a_2 x(t-2) = \varepsilon_t$$

Multiply eqn (1) by $x(t-h)$ and taking expectation on both sides we get

$$E[x(t).x(t-h)] + a_1 E[x(t-1)\,x(t-h)] + a_2 E[x(t-2)\,x(t-h)]$$

$$= E[\varepsilon_t\, x(t-h)]$$

which is called as Yule-Walker equation

for $h = 0$,

$$E[x(t).x(t)] + a_1 E[x(t-1).x(t)] + a_2 E[x(t-2)\,x(t)]$$

$$= E[\varepsilon_t\, x(t)]$$

$$\sigma_x^2 + a_1 \rho(1)\sigma_x^2 + a_2 \rho(2)\sigma_x^2 = \sigma^2$$

$$\sigma_x^2[1 + a_1\rho(1) + a_2\rho(2)] = \sigma^2$$

$$\Rightarrow \sigma_x^2 = \frac{\sigma^2}{1 + a_1\rho(1) + a_2\rho(2)} \quad \text{——} \quad (5)$$

for $h = 1$,

$$E[x(t)\,x(t-1)] + a_1 E[x(t-1)\,x(t-1)] + a_2 E[x(t-2)\,x(t-1)]$$

$$= E[\varepsilon_t\, x(t-1)]$$

$$\rho(1)\sigma_x^2 + a_1\rho(0)\sigma_x^2 + a_2\rho(1)\sigma_x^2 = 0$$

$$\{\because E[\varepsilon_t\, x(t-1)] = 0$$

$$\sigma_x^2[\rho(1) + a_1\rho(0) + a_2\rho(1)] = 0$$

$$\rho(1) + a_1\rho(0) + a_2\rho(1) = 0$$

$$\rho(1) + a_1 + a_2\,\rho(1) = 0 \qquad \{\because \rho(0) = 1\}$$

$$\rho(1)[1 + a_2] + a_1 = 0.$$

$$\rho(1)[1 + a_2] = -a_1$$

$$\rho(1) = \frac{-a_1}{1 + a_2}$$

for $h = 2$,

$$E[x(t)\,x(t-2)] + a_1 E[x(t-1)\,x(t-2)] + a_2 E[x(t-2)\,x(t-2)]$$

$$= E[\varepsilon_t\, x(t-2)]$$

$$\rho(2)\sigma_x^2 + a_1\,\rho(1)\sigma_x^2 + a_2\,\rho(0)\sigma_x^2 = 0.$$

$$\sigma_x^2[\rho(2) + a_1\rho(1) + a_2\,\rho(0)] = 0.$$

$$\rho(2) + a_1\rho(1) + a_2 = 0.$$

$$\rho(2) + a_1\left(\frac{-a_1}{1 + a_2}\right) + a_2 = 0.$$

$$\rho(2) - \frac{a_1^2}{1 + a_2} + a_2 = 0$$

$$\rho(2) = \frac{a_1^2}{1 + a_2} - a_2$$

substitute $\rho(1)$ & $\rho(2)$ in eq (5),

$$\sigma_x^2 = \frac{\sigma^2}{1 + a_1\left(\frac{-a_1}{1 + a_2}\right) + a_2\left(\frac{a_1^2}{1 + a_2} - a_2\right)}$$

Let us consider the denominator,

$$1 - \frac{a_1^2}{1+a_2} + \frac{a_1^2 a_2}{1+a_2} - a_2^2$$

$$= \frac{(1+a_2) - a_1^2 + a_1^2 a_2 - a_2^2(1+a_2)}{1+a_2}$$

$$= \frac{1+a_2 - a_1^2 + a_1^2 a_2 - a_2^2 + a_2^3}{1+a_2}$$

$$= \frac{(1+a_2)(1-a_2^2) - a_1^2(1-a_2)}{1+a_2}$$

$$= \frac{(1+a_2)(1+a_2)(1-a_2) - a_1^2(1-a_2)}{1+a_2}$$

$$= \frac{(1-a_2)\left[(1+a_2)(1+a_2) - a_1^2\right]}{1+a_2}$$

$$= \frac{(1-a_2)\left[(1+a_2)^2 - a_1^2\right]}{1+a_2}$$

$$= \frac{(1-a_2)\left[(1+a_2)+a_1\right]\left[1+a_2-a_1\right]}{1+a_2}$$

$$= \frac{(1-a_2)(1+a_1+a_2)(1-a_1+a_2)}{1+a_2}$$

$$\therefore \sigma_x^2 = \frac{\sigma^2(1+a_2)}{(1-a_2)(1+a_1+a_2)(1-a_1+a_2)}$$

In general, the Yule-Walker eqn/. are

$$\rho(k) + a_1 \rho(k-1) + a_2 \rho(k-2) = 0 \quad \text{for } k = 1, 2, 3 \ldots$$

solving this difference eqn for $\rho(k)$, we get

$$\rho(k) = \alpha_1 \mu_1^k + \alpha_2 \mu_2^k$$

where α_1 & α_2 are constants and they are determined by using $\rho(0) = 1$ & $\rho(1) = \frac{-a_1}{1+a_2}$

Thus for unequal roots i.e $(\mu_1 \neq \mu_2)$

$$\rho(k) = \frac{(1-\mu_2^2)\mu_1^{k+1} - (1-\mu_1^2)\mu_2^{k+1}}{(\mu_1 - \mu_2)(1+\mu_1\mu_2)} ; \; k = 0, 1, \ldots$$

Note:

when the roots are real and equal $(\mu = \mu_1 = \mu_2)$ then $\rho(k)$ can be written as,

$$\rho(k) = \left[1 + \frac{k(1-\mu^2)}{(1+\mu^2)}\right] \mu^k ; \; k = 0, 1, \ldots$$

$\Rightarrow$ The process is cov. stationary and the graph of the $\rho(k)$ is called as correlogram and it is oscillatory.

Auto-Regressive process of order k : AR(k)

The process $x(t)$ is said to be an Auto-Regressive process of order 'k', if it satisfies the difference eqn

$$x(t) + a_1 x(t-1) + a_2 x(t-2) + \cdots + a_k x(t-k) = \varepsilon_t \quad \text{——— (1)}$$

where $a_1, a_2, \ldots, a_k$ are all constants $\ni a_k \neq 0$ and ε_t is purely random process $\ni E(\varepsilon_t) = 0$; $V(\varepsilon_t) = \sigma^2$

eqn (1) can be rewritten as,

$$x(t) + a_1 B^1 x(t) + a_2 B^2 x(t) + \ldots + a_k B^k x(t) = \varepsilon_t$$

$$\left[1 + a_1 B + a_2 B^2 + \ldots + a_k B^k\right] x(t) = \varepsilon_t$$

$$\alpha(B)\, x(t) = \varepsilon_t \qquad \text{——— (2)}$$

where $\alpha(B) = \sum_{r=0}^{k} a_r B^r$; $a_0 = 1$ & B is a backward shift operator.

The soln. of the eqn (1) can be written as

$$x(t) = f(t) + \alpha^{-1}(B)\, \varepsilon_t \qquad \text{——— (3)}$$

$\therefore f(t) = A_1 \mu_1^t + A_2 \mu_2^t + \ldots + A_k \mu_k^t$ and $\mu_1, \mu_2, \ldots, \mu_k$ are all distinct roots of the polynomial

$$g(z) = z^k + a_1 z^{k-1} + \ldots + a_k$$

for the asymptotic stationarity of $x(t)$, all μ_k's must be less than 1 in ~~abs~~ absolute value.

Expanding the series $\alpha^{-1}(B) = \prod_{i=1}^{k} (1 - \mu_i B)^{-1}$ into partial fraction, the particular soln.

obtained as,

$$x(t) = \sum_{r=0}^{\infty} b_r' \, \varepsilon_{t-r}$$

where b_r' are constants involving μ_i's

Thus an auto-regressive process can be represented by an Moving Average process of infinite order.

Properties of covariance function & correlation function :-

(i) cov. fn. is even & +ve definite.

(ii) correlation fn. has the representation as

$$\rho(k) = \int_{-\pi}^{\pi} e^{ik\omega} \, dF_1(\omega)$$

where $F_1(\omega)$ is the dist. fn. and it determines a characteristic fn. uniquely.

when $F_1(\omega)$ is absolutely continuous then

$$f_1(\omega) = \frac{dF_1(\omega)}{d\omega}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \rho(k) \, e^{-ik\omega}$$

IIIly, the cov. fn. c_k can be represented as,

$$c_k = \int_{-\pi}^{\pi} e^{ik\omega} \, dF(\omega)$$

where $dF(\omega) = c_0 \, dF_1(\omega)$, when

$$dF(\omega) = f(\omega)\, d\omega \quad \text{and}$$

$$f(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} c_k e^{-ik\omega}$$

$\Rightarrow$ The fn. $F_1(\omega)$ & $F(\omega)$ are called integrated spectrum.

The fn $f(\omega)$ is called spectral density fn. and $f_1(\omega)$ is normalized spectral density fn. of the process.

Eg:

Let $x(t)$ be a purely random process with $E[x(t)] = 0$

$$E(x_r . x_s) = \begin{cases} \sigma^2 & ; r = s \\ 0 & ; r \neq s \end{cases}$$

$\Rightarrow c_k = 0$ for $k \neq 0$, $c_0 = \sigma^2$

$$\therefore f(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} c_k e^{-ik\omega}$$

$$= \frac{\sigma^2}{2\pi} \; ; \quad -\pi \leq \omega \leq \pi$$

Thus the fn. c_k may be obtained from the relation

$$c_k = \int_{-\pi}^{\pi} e^{ik\omega} \, dF(\omega)$$

Spectral density of time series (continuous parameter processes)

When the time parameters are continuous, the S.P $\{x(t), -\infty < t < \infty\}$ is wide-sense stationary iff $\forall$ 't'

$$E[x(t)] = m \text{ (say) is indep. of 't'}$$

cov. fn. $R(v) = E\{[x(t)-m][x(t+v)-m]\}$ is a fn. of 'v'

The cov. fn. $R(v)$ can be represented as

$$R(v) = \int_{-\infty}^{\infty} e^{iv\omega}\, dF(\omega)$$

where F is a dist. fn. & $dF(\omega) = f(\omega)d\omega$

The spectral density of the process is obtained as

$$f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iv\omega} R(v)\, dv$$

which is due to Bochner's inversion theorem.

Example:

Find the spectral density of the process $x(t)$ with cov. fn.

$$R(v) = a e^{-b|v|}$$

Soln.

Given that the cov. fn.

$$R(v) = a e^{-b|v|}$$

Then by using Bochner's inversion then the spectral density is obtained as

$$\phi(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iv\omega} R(v)\,dv$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iv\omega}\, a \cdot e^{-b|v|}\,dv$$

$$= \frac{a}{2\pi}\left[\int_{-\infty}^{0} e^{-iv\omega} e^{-b(-v)}\,dv + \int_{0}^{\infty} e^{-iv\omega} e^{-b(v)}\,dv\right]$$

$$= \frac{a}{2\pi}\left[\int_{-\infty}^{0} e^{(b-i\omega)v}\,dv + \int_{0}^{\infty} e^{-(b+i\omega)v}\,dv\right]$$

$$= \frac{a}{2\pi}\left[\left.\frac{e^{(b-i\omega)v}}{(b-i\omega)}\right|_{-\infty}^{0} + \left.\frac{e^{-(b+i\omega)v}}{-(b+i\omega)}\right|_{0}^{\infty}\right]$$

$$\left\{\because \int e^{2x} = \frac{e^{2x}}{2}\right\}$$

$$= \frac{a}{2\pi}\left[\frac{1}{b-i\omega} + \frac{1}{b+i\omega}\right]$$

$$= \frac{a}{2\pi}\left[\frac{b+\cancel{i\omega}+b-\cancel{i\omega}}{(b-i\omega)(b+i\omega)}\right]$$

$$= \frac{\cancel{2}ab}{\cancel{2}\pi}\left[\frac{1}{b^2+\omega^2}\right] \qquad \left\{\because b^2 - \cancel{i\omega b} + \cancel{i\omega b} - (i\omega)^2,\ i^2 = -1\right.$$

$$\therefore \phi(\omega) = \frac{ab}{\pi}\left[\frac{1}{b^2+\omega^2}\right]$$

Example :

Find the spectral density of the process when $R(v) = e^{-v^2/2}$

Soln

Given that the cov. fn as

$$R(v) = e^{-v^2/2}$$

By using Bochner's inversion thm, the spectral density is obtained as

$$\phi(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iv\omega} R(v)\,dv$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iv\omega} e^{-v^2/2}\,dv$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-1/2(v^2+2iv\omega)}\,dv$$

Add & subtract $(i\omega)^2$

$$\phi(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-1/2\left[v^2+2iv\omega+(i\omega)^2-(i\omega)^2\right]}\,dv$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-1/2(v+i\omega)^2} e^{-1/2\,\omega^2}\,dv$$

$$= \frac{1}{2\pi} e^{-1/2\,\omega^2}\int_{-\infty}^{\infty} e^{-1/2(v+i\omega)^2}\,dv$$

$$= \frac{1}{2\pi} e^{-1/2\,\omega^2}\sqrt{2\pi} \qquad \left\{\because \int_{-\infty}^{\infty} e^{-1/2(v+i\omega)^2}\,dv = \sqrt{2\pi}\right.$$

$$\phi(\omega) = \frac{1}{\sqrt{2\pi}} e^{-1/2\,\omega^2}$$